Generalized Network Externality Function

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Abstract

With regards to the network externality function, the first derivative is commonly considered to be positive. However, the sign of its second derivative is based on the assumption about the marginal network value when the network grows. Three key assumptions are constant, decreasing, or increasing which generate a different functional form, linear, concave, convex function, respectively. There is an effort to mix these strict assumptions by combining them which generate the S-shaped function such as logistic, Gompertz or Sigmoid function. Unfortunately, to do so, the S-shaped function causes an inappropriate domain, negative network size. In this work, we develop a dynamic model to generalize a functional form of network externality function which not only keeps all good properties of the S-shaped function, but also has a desired domain.

Keywords: Network externality, S-shaped function, Sigmoid function

JEL Classification:　　　　C51, D01, D11

1. Introduction

The network externality function (Katz and Shapiro, 1985) is the function that describes the relationship between network value and its corresponding size. This function is successfully utilized in the study of economics of network industries (Shy, 2001). We remark that the concept of network externality was introduced by Bell's employee, N.Lytkins (1917). Historically, the development of the function is centered on the assumption of properties. Let N be network size, $g: \mathbb{R} \rightarrow \mathbb{R}$ be the network externality function and be $g(N)$ the network externality value.

Because of the definition of network externality (Economides, 1996), network externality value is commonly assumed to increase when network size increases; that is, the first derivative of g is positive. Hence,

$$g'(N) > 0. \qquad (1.1)$$

Lately, the diminishing concept, in addition to the positive slope, is also normally assumed

$$g''(N) < 0 \text{ and } \lim_{N \to \infty} g'(N) = 0. \quad (1.2)$$

The properties in (1.2) supports that the network externality is the concave function. However, in general, Hans-Werner Gottinger (2003, p.17) said three key assumptions about the relationship between network size and network externality value relate to linear, logarithmic and exponential functional form. The linear function postulates that, as networks grow, the marginal value is constant. The logarithmic function postulates that, as a network grows, the marginal value diminishes. Network externalities at the limit in this formulation must be either negative or zero. The exponential function postulates that, as a network grows, the marginal value increases, which in the popular business and technology press, has been named 'Metcalfe's Law, Robert Metcalfe (1995). Moreover, an S-shaped function, in addition to these assumptions, is a mixture of an exponential function and a logarithm function; that is, the early additions to the network add exponentially, yet later additions add diminishingly in network externality value. The S-shaped function is an increasing function and has two horizontal asymptotes which are the appropriate properties of a network externality function. In addition to the S-shaped function, an N-shaped function is another mixture; that is, early additions add diminishing property, yet later additions add exponential property. In summary, the shape of a network externality function is based on the assumptions of particular goods. See Table 1.1.

Remark 1:

The example of an N-shaped network externality function is the product that develops itself when the network size reaches a certain level. The smartphones, for example, has few applications in the beginning in which diminishing concept in marginal network externality is applied. Later, when network size increases, there are more developers creating many new applications. At this point, the marginal network externality is no longer diminishing, but exponentially increasing.

Table 1.1   The characteristic of network externality function under various assumptions

Assumption	$g''(N)$	Shape
I	Zero	line
II	Negative	logarithm
III	Positive	exponential
IV	positive for the early growth, negative for late growth	S-shape
V	negative for the early growth, positive for late growth	N-shape

Next section, we develop a dynamic process to generalize a functional form of a network externality function which matches all assumptions and keeps all appropriate properties. The development of this function is motivated by Sigmoid function also called the Sigmoidal curve (von Seggern, 2007).

2. Network Externality Process

The linear, logarithm, exponential and S-shaped function can be considered the solution of corresponding differential equations. The well known S-shaped function is Sigmoid function, logistic function, Gompertz function and a cumulative distribution function. Sigmoid function is the solution of the differential equation.

$$dg = k(g - a)(b - g)dN \qquad (2.1)$$

where $k > 0$, $a < g < b$. The solution of (2.1) is

$$g(N) = b - \frac{b-a}{1+\left(\frac{g_0-a}{b-g_0}\right)e^{k(b-a)N}} \qquad (2.2)$$

where $g_0 = g(N_0)$. The equation (2.2) is increasing and differentiable function over domain, $(-\infty, \infty)$, and has two horizontal asymptotes,

$\lim_{N\to-\infty} g(N) = a$ and $\lim_{N\to\infty} g(N) = b$. This function meets the common properties of a network externality function (Katz and Shapiro, 1985), $g'(N) > 0$ and $\lim_{N\to\infty} g'(N) = 0$. However, its domain, representing the network size, is inappropriate because the network size cannot be negative. See Figure 2.1a. In this section, we introduce a differential equation in which its solution is the generalized network externality function that keeps all properties of an S-shaped function and eliminates the inappropriate domain. Let us define some terms.

Definition 2.1 Lower Limit of Network Externality Value is the greatest lower bound $(g.l.b)$ or the infimum of the range of a given network externality function, $\boldsymbol{inf}R_g$.

Definition 2.2 Upper Limit of Network Externality Value is the least upper bound $(l.u.b)$ or the supremum of the range of the network externality function, $\boldsymbol{sup}R_g$.

Thus, by itself, the network externality value can be explained by the Sigmoid differential equation

$$dg = k_g(g - a)(b - g)dt \qquad (2.3)$$

for $k_g > 0$ where $\boldsymbol{inf}R_g = a$ and $\boldsymbol{sup}R_g = b$.

Definition 2.3 Lower Limit of Network Size is the greatest lower bound or the infimum of the domain

of a given network externality function, $infD_g.$.

Definition 2.4 Upper Limit of Network Size is the least upper bound or the supremum of the domain of the network externality function, $supD_g.$

Like above, the network size can be explained by the Sigmoid differential equation

$$dN = k_N(N - c)(d - N)dt \qquad (2.4)$$

for $k_N > 0$ where $infD_g = c$ and $supD_g = d$.

Now, we are ready to introduce the concept of network externality process. The differential equation described by the ratio of the Sigmoid differential equation of network value (2.3) to the Sigmoid differential equation of network size (2.4) is called the network externality differential equation. All goods are called network goods if their marketability is influenced by the network externality process. Hence,

$$dg = k\frac{(g-a)(b-g)}{(N-c)(d-N)}dN \qquad (2.5)$$

where $k > 0$, $a < g(N) < b$ and $c < N < d$ is the network externality differential equation and its solution is
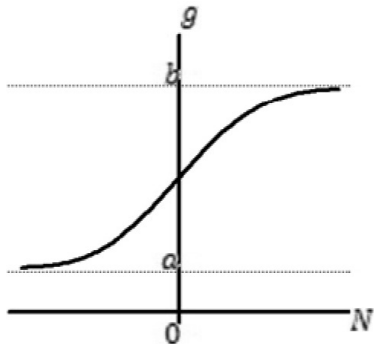
$$g(N) = b - \frac{b-a}{1+\left(\frac{g_0-a}{b-g_0}\right)\left(\frac{d-N_0}{N_0-c}\right)^{k\left(\frac{b-a}{d-c}\right)}\left(\frac{N-c}{d-N}\right)^{k\left(\frac{b-a}{d-c}\right)}}, \qquad (2.6)$$

where $g_0 = g(N_0)$. The equation (2.6) is called the generalized network externality function. For further graphical details, see Figure 2.1b.

Remark 2:

Without loss of generality, the rate of change, k, of a differential equation (2.5) can be relaxed to have a negative sign which implies the negative network externality.

Remark 3:

If we assume , $a = 0,$ $b = 1,$, and set $g(c) = 0$ and $g(d) = 1$, then the generalized network externality function fulfills the properties of a cumulative distribution function (CDF). In general, we can say that $\mu(n) = \frac{g(N)-a}{b-a}$ is a cumulative distribution function of a continuous random variable between the lower limit of network size and the upper limit of network size.
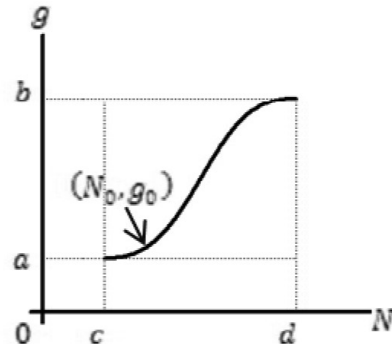

Figure 2.1a

Figure 2.1b

Figure 2.1 The Sigmoid and Generalized Network Externality Function

Before we discuss more about the properties and advantages of generalized network externality function, let first us define more technical terms.

Definition 2.5 The Lower-Left Terminal Point (LLTP) is a pair of the greatest lower bound of the domain and range of the function. Hence, the LLTP of generalized network externality function is $\left(inf D_g, inf R_g\right) = (c, a)$.

Definition 2.6 The Upper-Right Terminal Point (URTP) is a pair of the least upper bound of the domain and range of function. Hence, the URTP of generalized network externality function is $\left(sup D_g, sup R_g\right) = (d, b)$.

Let l be the line that connects these two points. Thus the equation of l is $l(N) = m(N - c) + a$ where $m = \frac{b-a}{d-c}$. Let $I = \left(\frac{g_0-a}{b-g_0}\right)\left(\frac{d-N_0}{N_0-c}\right) = \left(\frac{g_0-a}{N_0-c}\right)/\left(\frac{b-g_0}{d-N_0}\right)$. Let be the ratio of the slope of LLTP and the initial point to the slope of URTP and the initial point of the generalized network externality function.

Proposition 2.1. (see Figure 2.2) For $(N_0, g_0) \in (c, d) \times (a, b)$,

   i.   If (N_0, g_0) locates on line l, then $I = 1$

   ii.   If (N_0, g_0) locates under line l, then $I < 1$

   iii. If (N_0, g_0) locates above line l, then $I > 1$

Proof i. If (N_0, g_0) locates on the same line l as two terminal points, (c, a) and (d, b) the slope of any two points are equal. Thus, $\frac{g_0-a}{N_0-c} = \frac{b-g_0}{d-N_0}$ which implies $I = 1$

Proof ii. Let $\varepsilon > 0$ and (N_0, g_0) and locates on line l,. Hence, locates under line l and $I = \frac{\frac{g_0-\varepsilon-a}{N_0-c}}{\frac{b-g_0+\varepsilon}{d-N_0}} = \frac{\frac{g_0-a}{N_0-c} - \frac{\varepsilon}{N_0-c}}{\frac{b-g_0}{d-N_0} + \frac{\varepsilon}{d-N_0}} < \frac{\frac{g_0-a}{N_0-c}}{\frac{b-g_0}{d-N_0}} = 1$
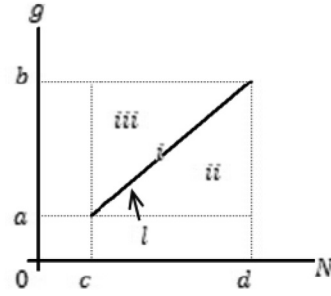
Proof iii. Let $\varepsilon > 0$ and (N_0, g_0) and locates on line l,. Hence, $(N_0, g_0 - \varepsilon) \in (c, d) \times (a, b)$ locates above line l and $I = \frac{\frac{g_0+\varepsilon-a}{N_0-c}}{\frac{b-g_0-\varepsilon}{d-N_0}} = \frac{\frac{g_0-a}{N_0-c} + \frac{\varepsilon}{N_0-c}}{\frac{b-g_0}{d-N_0} - \frac{\varepsilon}{d-N_0}} > \frac{\frac{g_0-a}{N_0-c}}{\frac{b-g_0}{d-N_0}} = 1$

Figure 2.2 The illustration for proposition 3.1

3. Properties of the Generalized Network Externality Function

In this section, we compare some properties of the network externality function under various assumptions to the generalized network externality function (2.6) and also its advantages. From Table 1.1, five assumptions about marginal network externality value when network grows are listed. Let us assign a particular function for the first four assumptions. To make them comparable, we also set the same domain and range for all functions (if possible).

Assumption I: $g_1(N) = \left(\frac{b-a}{d-c}\right)(N - c) + a$, where $c < N < d$

Assumption II: $g_2(N) = (b - a)\left(\frac{N-c}{d-c}\right)^{\theta} + a$, where $\theta < 1$ and $c < N < d$

Assumption III: $g_3(N) = (b - a)\left(\frac{N-c}{d-c}\right)^{\theta} + a$, where $\theta > 1$ and $c < N < d$

Assumption IV: g_4, , Sigmoid function (2.2)

3.1 Domain

In economics, the network size, domain of network externality function, is considered nonnegative value. For g_1, g_2 and g_3, and g_4, we can adjust them to get an appropriate domain by shifting, rescaling, and constraining the domain. The domain of g_4 is inappropriate, including the negative value, even though the shape of the Sigmoid function is more appropriate. If we shift or rescale its domain to a desired interval, we will drop the convergent property at the end behavior. The domain of the generalized network externality

function is $D_g = (c, d)$. This is an advantage, it not only has the desired domain but also saves all properties of Sigmoid function, especially the end behavior of both sides, $\lim_{N \to c^+} g(N) = a$ and $\lim_{N \to d^-} g(N) = b$.

3.2 Range

In economics, the network value, range of network externality function, is also considered nonnegative value. All function, $g_1 - g_4$ and generalized network externality function have the same range, $R_g = (a, b)$

3.3 Monotonicity

From the definition of network externality, with lower limit and upper limit , the positive marginal value of network value in (1.1) is expected. Again, all functions have this property. However, regarding to diminishing concept, the Sigmoid and generalized network externality function have this property. If we adjust the domain of Sigmoid function, we will lose this property. This is another advantage of the generalized network externality function; it has the diminishing property for both sides, $\lim_{N \to c^+} g'(N) = 0$ and $\lim_{N \to d^-} g'(N) = 0$.

3.4 Inflection Point

In economics, the inflection point of network externality affects the location of critical mass of the network. The functions, g_1, g_2 and g_3, and , have no critical point. The second derivative of Sigmoid function (2.2) is

$$g''(N) = 2g'(N)k\left(\frac{a+b}{2} - g(N)\right). \tag{3.1}$$

Thus, the Sigmoid function has only one critical point $\left(\frac{\ln\frac{b-g_0}{g_0-a}}{k(b-a)}, \frac{a+b}{2}\right)$, and the second derivative of generalized network externality function (2.6) is

$$g''(N) = \frac{2g'(N)}{(N-c)(d-N)}\left[k\left(\frac{a+b}{2} - g(N)\right) - \left(\frac{c+d}{2} - N\right)\right]. \tag{3.2}$$

It also has only one critical point (N_c, g_c) which satisfies the condition.

$$k\left(\frac{a+b}{2} - g(N)\right) = \left(\frac{c+d}{2} - N\right) \tag{3.3}$$

3.5 Concavity

The concavity of the network externality function is varying, based on the assumption about marginal network value of particular goods. The functions, g_1, g_2 and g_3, have zero, negative and positive concavity, respectively. The Sigmoid function has positive concavity in the early growth and negative in the later growth. This is another advantage of generalized network externality function, more flexible in concavity. It can provide linear, concave, convex, S-shaped and N-shaped function by controlling its parameters. The following are three scenarios which gives various shapes.

Scenario I: $k = \frac{1}{m}$

The rate of change of (2.5) equals to the reciprocal of the slope of l. With this condition, the equation (2.6) is reduced to

$$g(N) = b - \frac{b-a}{1+I\frac{N-c}{d-N}} = \frac{(bI-a)(N-d)+bI(d-c)}{(d-cI)+(I-1)N} \tag{3.4}$$

Proposition 3.1. For $k = \frac{1}{m}$ and $(N, g(N)) \in (c, d) \times (a, b)$,

i.　If $I = 1$, then $g''(N) = 0$ and $g(N) = l(N)$.

ii.　If $I < 1$, then $g''(N) > 0$ and $(N, g(N))$ locates under line l.

iii.　If $I > 1$, then $g''(N) < 0$ and $(N, g(N))$ locates above line l.

Proof　　Let $I = 1 + \varepsilon$. Thus, The equation (3.5) can be rewritten as

$$g(N) = \frac{(b-a)(N-d)}{(d-c)+(N-c)\varepsilon} + \frac{(d-c)b}{(d-c)+(N-c)\varepsilon} + \frac{(N-c)b\varepsilon}{(d-c)+(N-c)\varepsilon} \tag{3.5}$$

$$g(N) - l(N) = \frac{m(d-N)(N-c)\varepsilon}{(d-c)+(N-c)\varepsilon} \tag{3.6}$$

From (3.2), the second derivative of generalized network externality function can be rewritten as

$$g''(N) = \frac{-2g'(N)}{(N-c)(d-N)}k[g(N) - l(N)]. \tag{3.7}$$

Plug (3.6) into (3.7), we will get

$$g''(N) = \frac{-2mg'(N)\varepsilon}{(d-c)+(N-c)\varepsilon}. \qquad (3.8)$$

Hence, from (3.6) and (3.8),
If $I = 1$ then $\varepsilon = 0$, which $g(N) = l(N)$ and $g''(N) = 0$.
If $I < 1$ then $\varepsilon < 0$, which $g(N) < l(N)$ and $g''(N) > 0$.
If $I > 1$ then $\varepsilon > 0$, which $g(N) > l(N)$ and $g''(N) < 0$.

From proposition 3.2, when $k = \frac{1}{m}$ and $I = 1$, the equation (2.6) is reduced to linear function. In other words, g_1 is the special case of the generalized network externality function. When $k = \frac{1}{m}$ and $I < 1$, the generalized network externality function meets the increasing marginal network externality value assumption (assumption III) and when $k = \frac{1}{m}$ and $I > 1$, it meets the decreasing marginal network value assumption (assumption II). See Table 3.1.

Scenario II: $k > \frac{1}{m}$

For this scenario, the generalized network externality function is S-shaped and also preserves all properties of the Sigmoid function. See Table 3.1.

Scenario III: $k < \frac{1}{m}$

For this scenario, the generalized network externality function is N-shaped. See Table 3.1.

In summary, the followings are the advantages of the generalized network externality function

i. It has an appropriate domain.

ii. It preserves all good properties of the Sigmoid function.

iii. It meets all assumption in Table 1.1.

Table 3.1 The generalized network externality function in some condition of parameters

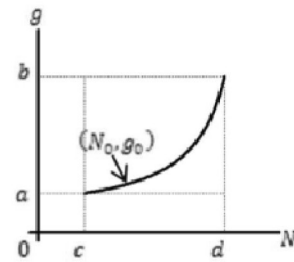| Property	Assumption					
	I	II	III	IV	IV	V
g	$k = 1/m$			N/A	$k > \dfrac{1}{m}$	$k < \dfrac{1}{m}$
	$I = 1$	$I > 1$	$I < 1$			
function	g_1	g_2	g_3	g_4		
D_g	(c,d)	(c,d)	(c,d)	$(-\infty,\infty)$	(c,d)	(c,d)
R_g	(a,b)	(a,b)	(a,b)	(a,b)	(a,b)	(a,b)
$g'(N)$	+	+	+	+	+	+
$g''(N)$	0	-	+	+ then -	+ then -	- then +
Shape	Linear	concave	convex	S-shaped	S-shaped	N-shaped
Graph	3.1a	3.1b	3.1c	3.1d	3.1e	3.1f

iv.
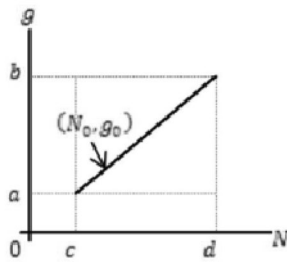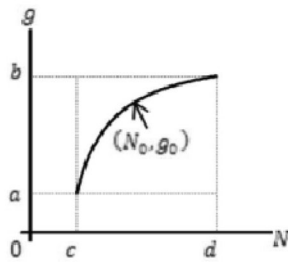
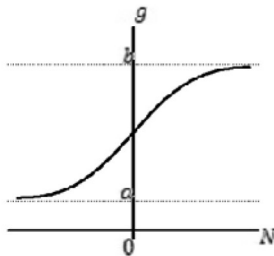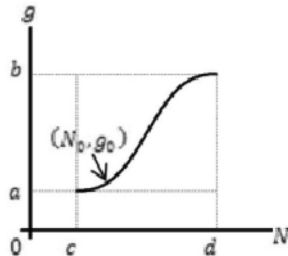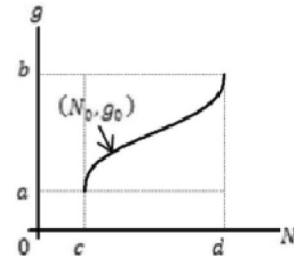| Figure 3.1a | Figure 3.1b | Figure 3.1c |

| Figure 3.1d | Figure 3.1e | Figure 3.1f |

V.   **Figure 3.1** The illustration of network externality function

4. Conclusion

Including the combination of assumptions, there are five assumptions about the marginal network externality value when network size increases. They generate different functional forms. Unfortunately, each function has some disadvantages. The functions, g_1, g_2 and g_3, do not have the diminishing concept and the shape of function is too strict. Even though the Sigmoid function, g_4, has more flexible shape and meets common properties of network externality function, it has an inappropriate domain. If we fix its domain to the desired interval, we will lose some good properties. By implying concept of differential equations, this work introduces the new function, called the generalized network externality function, which not only preserves all properties of Sigmoid function but also fix the inappropriate domain to the adjusted desired domain. Moreover, all network functions under five assumptions can be replaced by this generalized network externality function. In the study of its properties, the rate of change and the initial point of generalized network differential equation play the important role in controlling the shape of generalized network externality function.

5. Reference

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